THE ANNIHILATION THEOREM FOR THE COMPLETELY REDUCIBLE LIE SUPERALGEBRAS

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Abstract. A well known theorem of Duflo claims that the annihilator of a Verma module in the enveloping algebra of a complex semisimple Lie algebra is generated by its intersection with the centre. For a Lie superalgebra this result fails to be true. For instance, in the case of the orthosymplectic Lie superalgebra $\mathrm{osp}(1,2)$, Pinczon gave in [Pi] an example of a Verma module whose annihilator is not generated by its intersection with the centre of universal enveloping algebra. More generally, Musson produced in [Mu1] a family of such "singular" Verma modules for $\mathrm{osp}(1,2l)$ cases.

In this article we give a necessary and sufficient condition on the highest weight of a $\mathrm{osp}(1,2l)$-Verma module for its annihilator to be generated by its intersection with the centre. This answers a question of Musson.

The classical proof of the Duflo theorem is based on a deep result of Kostant which uses some delicate algebraic geometry reasonings. Unfortunately these arguments can not be reproduced in the quantum and super cases. This obstruction forced Joseph and Letzter, in their work on the quantum case (see [JL]), to find an alternative approach to the Duflo theorem. Following their ideas, we compute the factorization of the Parthasarathy–Ranga-Rao–Varadarajan (PRV) determinants. Comparing it with the factorization of Shapovalov determinants we find, unlike to the classical and quantum cases, that the PRV determinant contains some extrafactors. The set of zeroes of these extrafactors is precisely the set of highest weights of Verma modules whose annihilators are not generated by their intersection with the centre. We also find an analogue of Hesselink formula (see [He]) giving the multiplicity of every simple finite dimensional module in the graded component of the harmonic space in the symmetric algebra.

1. Introduction

The aim of this article is to study the generalization of the annihilator theorem of Duflo to a certain class of Lie superalgebras, namely to the class of orthosymplectic Lie superalgebras $\mathrm{osp}(1,2l)$, $l \geq 1$. The main reason for considering this class of Lie superalgebras is the complete reducibility of finite dimensional modules. Both in classical and quantum cases, the complete reducibility appears in various steps of the study of annihilators of Verma modules. A Lie superalgebra whose finite dimensional modules are completely reducible is called completely reducible. According to the theorem of

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Djoković and Hochschild (see [Sch], p. 239), any finite dimensional completely reducible Lie superalgebra is a direct sum of semisimple Lie algebras and algebras $\mathrm{osp}(1,2l)$ $(l \geq 1)$. Another useful property of $\mathrm{osp}(1,2l)$ is that its enveloping algebra is a domain (see [AL]).

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra $\mathrm{osp}(1,2l)$, and $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}_0 \simeq \mathrm{sp}(2l)$. Let $\Delta = \Delta_0 \cup \Delta_1$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, where Δ_0 (resp., Δ_1) is the set of even (resp., odd) roots. Let π be a set of simple roots of Δ and $P^+(\pi)$ be the set of integral dominant weights. Denote by Δ^+ (resp., Δ_0^+, Δ_1^+) the set of positive roots (resp., positive even, positive odd). Set $\overline{\Delta_0^+} = \Delta_0^+ \setminus 2\Delta_1^+$, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha - \frac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha$. Denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, by $\mathcal{Z}(\mathfrak{g})$ its centre. Let $\mathcal{F}$ be the canonical filtration of $\mathcal{U}(\mathfrak{g})$.

Consider the case $l = 1$, i.e. $\mathfrak{g} = \mathrm{osp}(1,2)$, which has been treated by Pinczon. In this case, any $\mathfrak{g}$-Verma module $\widetilde{M}$, viewed as a $\mathfrak{g}_0$-module, is the direct sum of two $\mathfrak{g}_0$-Verma modules M_0 and M_1. Let C_0 be a Casimir element for $\mathfrak{g}_0$. Then C_0 acts by scalars c_i on M_i (i=0,1). In general, $c_1 \neq c_0$, and in this case, Pinczon proved that $\mathrm{Ann}\,\widetilde{M} = \mathcal{U}(\mathfrak{g})\,\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}$. However when the highest weight of $\widetilde{M}$ equals $-\rho$, one has $c_1 = c_0$ and so $C_0 - c_0$ belongs to the annihilator of $\widetilde{M}$. It is easy to check that $(C_0 - c_0) \notin \mathcal{U}(\mathfrak{g})\,\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}$. Consequently, the annilihilator theorem doesn't hold in full generality.

Return to the general case $\mathfrak{g} = \mathrm{osp}(1,2l)$. For any $\lambda \in \mathfrak{h}^*$, let $\widetilde{M}(\lambda)$ be the Verma module of highest weight λ. Musson showed (see [Mu1],5.6) that if $(\lambda+\rho,\beta) = 0$ for some $\beta \in \Delta_1^+$ and $(\lambda+\rho,\alpha) \neq 0$ for all $\alpha \in \overline{\Delta_0^+}$ then $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}\widetilde{M}(\lambda)$ strictly contains $\mathcal{U}(\mathfrak{g})\,\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}(\lambda)$ and so the annihilator theorem does not hold. At the end of the article [Mu1], Musson asked for which $\lambda \in \mathfrak{h}^*$ the annihilator of $\widetilde{M}(\lambda)$ is generated by its intersection with the centre.

In this article we give an answer to this question by proving the following theorem:

Theorem. *For any $\lambda \in \mathfrak{h}^*$ one has*

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}\widetilde{M}(\lambda) = \mathcal{U}(\mathfrak{g})\,\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}(\lambda) \iff \forall\alpha \in \Delta_1^+ :\ (\lambda+\rho,\alpha) \neq 0. \tag{1}$$

Our strategy to prove this theorem is based on a recent alternative proof (see [J1]) of the classical annihilator theorem which follows essentially the quantum case treated by Joseph and Letzter— see [JL]. This proof has the following main steps.

1) The first ingredient is the separation theorem established by Musson in [Mu1], 1.4. Using complete reducibility, he proved the existence of an $ad\,\mathfrak{g}$-submodule $\mathcal{H}$ of $\mathcal{U}(\mathfrak{g})$ such that the multiplication map induces an isomorphism $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{Z}(\mathfrak{g}) \otimes \mathcal{H}$ of $ad\,\mathfrak{g}$-modules. Moreover, the multiplicity of any finite dimensional module in $\mathcal{H}$ is equal to the dimension of its zero weight space. Since the centre acts on any Verma module $\widetilde{M}(\mu)$ as a scalar, the separation theorem implies that $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}\widetilde{M}(\mu) = \mathcal{U}(\mathfrak{g})\,\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}(\mu)$ iff $\mathrm{Ann}_{\mathcal{H}}\widetilde{M}(\mu) = 0$.

2) In Section 5 we obtain a formula giving the multiplicity of every simple finite dimensional module $\widetilde{V}(\lambda)$, $\lambda \in P^+(\pi)$, in the graded components H^n where $H = gr_{\mathcal{F}}\mathcal{H}$. This is an analogue of the classical Hesselink formula (see [He]). However, in our case the formula looks rather suprising. Namely, for every $\lambda \in P^+(\pi)$, the multiplicity of every $\widetilde{V}(\lambda)$ in $\mathcal{H}$ is given by the formula

$$[H^n : \widetilde{V}(\lambda)] = \sum_{w\in W} (-1)^{l(w)} P_n(w.\lambda), \tag{2}$$

where $P_n(\mu)$ are integers defined by generating function

$$\prod_{\alpha\in\Delta_0^+} (1 - qe^{\alpha})^{-1}(1 + q^{2l}e^{\beta_1}) \prod_{\beta\in\Delta_1^+\setminus\{\beta_1\}} (1 + e^{\beta}) = \sum_{r=0}^{\infty} \sum_{\nu\in\mathbb{N}\pi} P_r(\nu)e^{\nu}q^r \tag{3}$$

where β_1 is the highest weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$.

3) Any Verma module $\widetilde{M}(\mu)$ has a simple Verma submodule $\widetilde{M}(\mu')$ and $\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}(\mu) = \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}(\mu')$. Moreover, $(\mu + \rho, \alpha) = 0$ for some $\alpha \in \Delta_1^+$ iff $(\mu' + \rho, \alpha') = 0$ for some $\alpha' \in \Delta_1^+$. Thus, in order to prove the implication "$\Leftarrow$" of (1), it is sufficient to check it for simple Verma modules.

For the other implication, we show (see 7.2) that there exists an $ad\,\mathfrak{g}$ submodule V of $\mathcal{U}(\mathfrak{g})$, which lies in the annihilator of any simple Verma module $\widetilde{M}(\mu)$ such that $(\mu+\rho, \alpha) = 0$ for some $\alpha \in \Delta_1^+$. Then, Lemma 7.3 implies that V lies indeed in the annihilator of any Verma module $\widetilde{M}(\mu)$ such that $(\mu + \rho, \alpha) = 0$ for some $\alpha \in \Delta_1^+$. So in a sense, both implications of the equivalence (1) are reduced to the case of a simple Verma module.

4) In Section 4 we present a criterion of simplicity for a Verma module $\widetilde{M}(\mu)$ given by Kac (see [K3] and also [Mu2] 2.4). As in the classical case, this criterion is related to the so-called Shapovalov determinants $\det S_\nu \in \mathcal{U}(\mathfrak{h})$, $\nu \in \mathbb{N}\pi$. These determinants are constructed in such a way that $\det S_\nu(\mu) = 0$ means that the Verma module $\widetilde{M}(\mu)$ contains a proper submodule with a non trivial element of weight $\mu - \nu$. Thus, $\widetilde{M}(\mu)$ is simple iff $\det S_\nu(\mu) \neq 0$ for all $\nu \in \mathbb{N}\pi$. Kac proved that all factors of Shapovalov determinants are linear and he gave an explicit formula for the factorization of these determinants (see 4.2).

5) The separation theorem allows one to define the Parthasarathy–Ranga-Rao–Varadarajan (PRV) determinants as in the classical case (see 6.2 for details). For any $\lambda \in \mathfrak{h}^*$, denote by $\widetilde{V}(\lambda)$ the unique simple module of the highest weight λ. The PRV determinant $\det PRV^\lambda \in \mathcal{U}(\mathfrak{h})$, $\lambda \in P^+(\pi)$, has the property:

$$\forall\mu \in \mathfrak{h}^*,\ \det PRV^\lambda(\mu) = 0 \Longleftrightarrow \exists\widetilde{V}(\lambda) \subset \mathcal{H}, \widetilde{V}(\lambda) \subset \mathrm{Ann}_{\mathcal{H}}\widetilde{V}(\mu).$$

Thus $\mathrm{Ann}_{\mathcal{H}}\widetilde{V}(\mu) = 0$ iff $\det PRV^\lambda(\mu) \neq 0$ for all $\lambda \in P^+(\pi)$. Therefore we have to prove that for any simple Verma module $\widetilde{M}(\mu) = \widetilde{V}(\mu)$ one has the following equivalence

$$\exists\lambda \in P^+(\pi) :\ \det PRV^\lambda(\mu) = 0 \quad\Longleftrightarrow\quad \exists\alpha \in \Delta_1^+ :\ (\mu + \rho, \alpha) = 0. \tag{4}$$

In Section 6 we describe the zeroes of the determinants $\det PRV^\lambda$.

Both in the classical semisimple and quantum cases the union of the zeroes of $\det PRV^\lambda$ $(\lambda \in P^+(\pi))$ coincides with the union of zeroes of the Shapovalov determinants $\det S_\nu (\nu \in \mathbb{N}\pi)$. This implies the annihilation theorem.

6) In Theorem 6.5 we give a linear factorization of the determinants $\det PRV^\lambda : \ \lambda \in P^+(\pi)$. There are factors of two types. The factors of the first type, called "standard factors", coincide with factors of Shapovalov determinants. They can be obtained as reviewed in [J1]. We briefly summarize this procedure in 6.6— 6.7. These factors have form

$$\begin{array}{lll} \varphi(\alpha) + (\rho, \alpha) - m(\alpha,\alpha)/2, & \alpha \in \overline{\Delta_0^+}, & m \in \mathbb{N}, m \geq 1 \\ \varphi(\alpha) + (\rho, \alpha) - (2m+1)(\alpha,\alpha)/2, & \alpha \in \Delta_1^+, & m \in \mathbb{N} \end{array} \tag{5}$$

where the element $\varphi(\alpha)$ of $\mathfrak{h}$ is defined by the formula $\varphi(\alpha)(\mu) = (\alpha, \mu)$.

A delicate point is to get the factors of the second type, called "exotic factors". They have form

$$\varphi(\alpha) + (\rho, \alpha), \ \alpha \in \Delta_1^+. \tag{6}$$

The Hesselink formula discussed in 2) ensures that factors (5), (6) indeed exhaust the set of factors of PRV determinants.

The exotics factors correspond precisely to the equivalence (4). We conclude that the annihilator of a simple Verma module $\widetilde{M}(\mu) = \widetilde{V}(\mu)$ is generated by its intersection with the centre $\mathcal{Z}(\mathfrak{g})$ iff μ is not a zero of an exotic factors of some PRV determinant.

Remark. Taking into account that any completely reducible Lie superalgebra is a direct sum of simple Lie algebras and Lie superalgebras of the type $\mathrm{osp}(1,2l)$, our results (theorems 5.4, 6.5, 7.1) can be directly extended to the case of any finite dimensional completely reducible Lie superalgebra $\mathfrak{g}$.

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2. Background

In this Section we fix the main notations we use throughout this paper. The notation $\mathbb{N}^+$ will stand for the set of positive integers. The base field we are going to work with is $\mathbb{C}$.

2.1. Let $\mathfrak{g}$ be the Lie superalgebra $\mathrm{osp}(1,2l),\ l \geq 1$ (see Kac [K2] for a presentation of this Lie superalgebra by generators and relations). Denote by $\mathfrak{g}_0$ the even part and by $\mathfrak{g}_1$

the odd part of $\mathfrak{g}$. We recall that $\mathfrak{g}_0 \simeq \mathrm{sp}(2l)$. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_0$. Denote by Δ_0 (resp., Δ_1) the set of even (resp., odd) roots of $\mathfrak{g}$. Set $\Delta = \Delta_0 \cup \Delta_1$. Let Δ_{irr} be the set of irreducible roots of Δ. Then $\Delta_{irr} = \overline{\Delta_0} \cup \Delta_1$, where $\overline{\Delta_0}:=\Delta_0 \backslash 2\Delta_1$.

Fix a basis of simple roots π of Δ, and define correspondingly the sets $\Delta^{\pm}, \Delta_0^{\pm}, \Delta_1^{\pm}, \overline{\Delta_0}^{\pm}, \Delta_{irr}^{\pm}$. Denote by W the Weyl group of Δ. Set

$$\rho_0 := \tfrac{1}{2} \sum_{\alpha\in\Delta_0^+} \alpha, \quad \rho_1 := \tfrac{1}{2} \sum_{\alpha\in\Delta_1^+} \alpha, \quad \rho := \rho_0 - \rho_1 = \tfrac{1}{2} \sum_{\alpha\in\Delta_{irr}^+} \alpha.$$

Introduce the standard partial order relation on $\mathfrak{h}^*$: $\lambda \le \mu \Longleftrightarrow \mu - \lambda \in \mathbb{N}\pi$.

Denote by $(-,-)$ the non-degenerate bilinear form on $\mathfrak{h}^*$ coming from the restriction of the Killing form of $\mathfrak{g}_0$ to $\mathfrak{h}$. Let $\varphi : \mathfrak{h}^* \longrightarrow \mathfrak{h}$ be the isomorphism given by $\varphi(\lambda)(\mu) := (\lambda, \mu)$. For any $\lambda, \mu \in \mathfrak{h}^*,\ \mu \neq 0$ one defines

$$\langle \lambda, \mu \rangle := 2\frac{(\lambda,\mu)}{(\mu,\mu)}.$$

2.2. One has the following useful realization of Δ. Identify $\mathfrak{h}^*$ with $\mathbb{C}^l$ and consider $(-,-)$ as a scalar product on $\mathbb{C}^l$. Then there exists an orthonormal basis $\{\beta_1, \dots, \beta_l\}$ such that

$$\begin{array}{ll}
\pi = \{\beta_1 - \beta_2, \dots, \beta_{l-1} - \beta_l, \beta_l\} & \\
\Delta_0^+ = \{\beta_i \pm \beta_j, 1 \le i < j \le l,\ 2\beta_i,\ 1 \le i \le l\}, & \Delta_1^+ = \{\beta_i, 1 \le i \le l\} \\
\Delta_{irr}^+ = \{\beta_i \pm \beta_j,\ \beta_i,\ 1 \le i < j \le l\}, & \overline{\Delta_0}^+ = \{\beta_i \pm \beta_j,\ 1 \le i < j \le l\} \\
\rho = \displaystyle\sum_{i=1}^{l} (l - i + \frac{1}{2})\beta_i, & \rho_0 = \displaystyle\sum_{i=1}^{l} (l - i + 1)\beta_i
\end{array}$$

and the Weyl group W is just the group of the signed permutations of the β_i. Define the translated action of W on $\mathfrak{h}^*$ by the formula: $w.\lambda = w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*$ and $w \in W$. For any element $w \in W$ set $\mathrm{sgn}\, w := (-1)^{l(w)}$, where $l(w)$ is the length of a reduced decomposition of w.

There exists a Chevalley antiautomorphism σ of $\mathfrak{g}$ of order 4 which leaves invariant the elements of $\mathfrak{h}$ and maps $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ for any $\alpha \in \Delta$.

2.3. **Enveloping algebra.** As usual, if $\mathfrak{k}$ is a Lie superalgebra, $\mathcal{U}(\mathfrak{k})$ denotes its enveloping algebra. Set $\mathcal{F}$ the natural filtration of $\mathcal{U}(\mathfrak{g})$ defined by $\mathcal{F} = (\mathfrak{g}^n)_{n\in\mathbb{N}}$. The graded algebra of $\mathcal{U}(\mathfrak{g})$ associated to $\mathcal{F}$ is the symmetric superalgebra denoted by $\mathcal{S}(\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}_0) \otimes \bigwedge \mathfrak{g}_1$ which is not a domain. Nevertheless, Aubry and Lemaire showed ([AL]) that $\mathcal{U}(\mathfrak{g})$ is a domain.

We define the supercentre to be the vector subspace of $\mathcal{U}(\mathfrak{g})$ generated by the homogenous elements a such that $ax = (-1)^{|a||x|}xa$ for all homogenous elements x in $\mathcal{U}(\mathfrak{g})$. For $\mathfrak{g} = \mathrm{osp}(1, 2l)$, the supercentre coincides with the genuine centre.

The Lie superalgebra $\mathfrak{g}$ acts on $\mathcal{U}(\mathfrak{g})$ and $\mathcal{S}(\mathfrak{g})$ by superderivation via the adjoint action. We denote these actions by *ad*. Throughout this paper, an action of any element of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ means always the adjoint action.

We identify $\mathcal{U}(\mathfrak{h})$ with $\mathcal{S}(\mathfrak{h})$.

2.4. **Verma and simple highest weight modules.** For any $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the one-dimensional subspace of weight α of $\mathfrak{g}$ and $\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha$, $\mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm$.

For a fixed $\lambda \in \mathfrak{h}^*$, let $\widetilde{\mathbb{C}}_\lambda$ be the one dimensional $\mathfrak{b}^+$-module with $\mathfrak{n}^+ v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$ and $v \in \widetilde{\mathbb{C}}_\lambda$. Set

$$\widetilde{M}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \widetilde{\mathbb{C}}_\lambda.$$

The Verma module $\widetilde{M}(\lambda)$ has a unique simple quotient denoted by $\widetilde{V}(\lambda)$.

2.5. **The $\widetilde{\mathcal{O}}$ category.** Let M be a $\mathfrak{g}$-module. For any $\lambda \in \mathfrak{h}^*$, set

$$M_\lambda = \{m \in M | \ hm = \lambda(h)m, \ \forall h \in \mathfrak{h}\}.$$

A non-zero vector $v \in M$ has weight λ if $v \in M_\lambda$. For any subspace N of M we denote by $\Omega(N)$ the set of weights $\lambda \in \mathfrak{h}^*$ such that $N \cap M_\lambda \neq \{0\}$. The module M is said to be diagonalizable if $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$. If M is a diagonalizable module and $\dim M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$, we set $\operatorname{ch} M = \sum_{\lambda\in\mathfrak{h}^*} \dim M_\lambda e^\lambda$.

Throughout this article, we shall work essentially with two categories of $\mathfrak{g}$-modules. The first one is the category of completely reducible $\mathfrak{g}$-modules. If M is such a module and V is any simple module, we shall denote by $[M : V]$ the number $\dim \operatorname{Hom}_{\mathfrak{g}}(V, M)$. The second one, denoted by $\widetilde{\mathcal{O}}$, is the full subcategory of the category of $\mathfrak{g}$-modules whose objects are $\mathfrak{g}$-modules M such that

(i) M is diagonalizable
(ii) $\forall \lambda \in \Omega(M), \dim M_\lambda < +\infty$
(iii) $\exists \lambda_1, \dots, \lambda_k \in \mathfrak{h}^* | \ \Omega(M) \subset \bigcup_{i=1,\dots,k} (\lambda_i - \mathbb{N}\pi)$

Any object of the $\widetilde{\mathcal{O}}$ category, considered as a $\mathfrak{g}_0$-module, belongs to the $\mathcal{O}$ category (see [J1], 3.5 for definition). In particular, any $\mathfrak{g}$-module M of $\widetilde{\mathcal{O}}$ has finite length. If V is a simple highest weight module, we denote by $[M : V]$ the number of simple quotients isomorphic to V in any composition series of M. This number does not depend of the choice of the composition series of M.

2.6. **Finite dimensional representations.** Define for $r \in \{1, \dots, l\}$ the fundamental weight $\omega_r = \sum_{i=1}^{r} \beta_i$, and introduce the set

$$P^+(\pi) := \sum_{r=1}^{l} \mathbb{N}\omega_r = \{\lambda \in \mathfrak{h}^* | \ \langle \lambda, \beta_l \rangle \in 2\mathbb{N}, \ \langle \lambda, \beta_i - \beta_{i+1} \rangle \in \mathbb{N}, \ \forall i = 1, \dots, l-1\}.$$

Kac (see [K1]) showed that $\widetilde{V}(\lambda)$ is finite dimensional iff $\lambda \in P^+(\pi)$. For any $\mu \in \mathfrak{h}^*$, let $V(\mu)$ be the simple $\mathfrak{g}_0$-module of highest weight μ. Remark that $\{\beta_1 - \beta_2, \dots, \beta_{l-1} - \beta_l, 2\beta_l\}$ is a basis of simple roots of Δ_0 and that $\langle \mu, 2\beta_l \rangle = \frac{1}{2}\langle \mu, \beta_l \rangle$ for all $\mu \in \mathfrak{h}^*$. Thus $V(\lambda)$ is finite dimensional iff $\lambda \in P^+(\pi)$.

2.6.1. Observe that the longest element of the Weyl group acts on $\mathfrak{h}^*$ as $-id$. Therefore for any finite dimensional module V, $\Omega(V) = -\Omega(V)$.

3. The separation theorem

Recall the separation theorem established by Musson in [Mu1],1.4:

3.1. **Theorem.** *There exists an ad-invariant subspace $\mathcal{H}$ in $\mathcal{U}(\mathfrak{g})$ such that the multiplication map induces an* $\operatorname{ad}\mathfrak{g}$*-isomorphism $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{Z}(\mathfrak{g}) \otimes \mathcal{H}$. Moreover, for every simple finite dimensional module $\widetilde{V}$, $[\mathcal{H} : \widetilde{V}] = \dim \widetilde{V}_0$.*

3.2. Since the centre $\mathcal{Z}(\mathfrak{g})$ acts by a scalar on every Verma module $\widetilde{M}(\mu)$, the separation theorem implies the following equivalence :

$$Ann_{\mathcal{U}(\mathfrak{g})}\widetilde{M}(\mu) = \mathcal{U}(\mathfrak{g}) \operatorname{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu) \iff \operatorname{Ann}_{\mathcal{H}} \widetilde{M}(\mu) = \{0\}.$$

As in the classical case, any Verma module contains a simple Verma submodule. Let $\widetilde{M}(\mu)$ be a Verma module and M be its simple Verma submodule. Obviously,

$$\operatorname{Ann}_{\mathcal{H}} \widetilde{M}(\mu) \subset \operatorname{Ann}_{\mathcal{H}} M.$$

Since $\widetilde{M}(\mu)$ and M have the same central character, M has the form $M \simeq M(w.\mu)$ for some $w \in W$ (see [Mu1],1.1). By definition of the translated action of W, $w.\mu + \rho = w(\mu + \rho)$ and as W acts on Δ_1 by signed permutations, one has

$$\exists \alpha \in \Delta_1^+, \ (\mu + \rho, \alpha) = 0 \iff \exists \alpha \in \Delta_1^+, \ (w.\mu + \rho, \alpha) = 0.$$

Hence the proof of the implication "$\Leftarrow$" of our main Theorem 7.1 can be reduced to the case of simple Verma modules. Indeed for this, it is enough to show that

$$\left[\widetilde{M}(\mu) \text{ simple and } (\mu + \rho, \alpha) \neq 0 \ \forall \alpha \in \Delta_1^+\right] \implies \operatorname{Ann}_{\mathcal{H}} \widetilde{M}(\mu) = 0. \tag{7}$$

4. Factorization of the Shapovalov determinants

4.1. The triangular decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^+)$ leads to the following decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+)$$

The projection $\Upsilon : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ with respect to this decomposition is called the Harish-Chandra projection. Observe that $\Upsilon(ha) = h\Upsilon(a)$ for all $h \in \mathcal{U}(\mathfrak{h}), a \in \mathcal{U}(\mathfrak{g})$.

The Harish-Chandra projection map allows us to define a contravariant form on each Verma module $\widetilde{M}(\lambda)$ by the formula

$$\langle av_\lambda, bv_\lambda \rangle_\lambda = \Upsilon(\sigma(a)b)(\lambda) \quad \forall a, b \in \mathcal{U}(\mathfrak{g}),$$

where v_λ is a primitive vector of highest weight λ.

The kernel of this form is the largest proper submodule of $\widetilde{M}(\lambda)$. Consequently, the Verma module $\widetilde{M}(\lambda)$ is simple if and only if the form $\langle -, - \rangle_\lambda$ is non-degenerate. The subspaces of different weights are pairwise orthogonal with respect to $\langle -, - \rangle_\lambda$. Therefore, the quest for a criterion of simplicity for a Verma module leads naturally to the analysis of the zeroes of the so called Shapovalov determinants: $\det S_\nu$, where $\nu \in \mathbb{N}\pi$ and $S_\nu : \mathcal{U}(\mathfrak{n}_-)_\nu \times \mathcal{U}(\mathfrak{n}_-)_\nu \mapsto \mathcal{S}(\mathfrak{h})$ is defined by the formula

$$S_\nu(x, y) = \Upsilon(\sigma(x)y).$$

The factorization of these determinants was established by Kac (see [K3]) for all classical simple Lie superalgebras. For the present case $\mathfrak{g} = \mathrm{osp}(1, 2l)$ (see also [Mu2] 2.4) one has the

4.2. **Theorem.** *For all $\nu \in \mathbb{N}$, one has, up to a non-zero scalar,*

$$\det S_\nu = \prod_{\alpha \in \overline{\Delta_0^+}} \prod_{m=1}^{\infty} \left(\varphi(\alpha) + (\rho, \alpha) - \frac{1}{2} m(\alpha, \alpha)\right)^{\tau(\nu - m\alpha)} \times$$
$$\prod_{\alpha \in \Delta_1^+} \prod_{m=1}^{\infty} \left(\varphi(\alpha) + (\rho, \alpha) - \frac{1}{2}(2m-1)(\alpha, \alpha)\right)^{\tau(\nu - (2m-1)\alpha)}$$

where $\tau : \mathbb{Z}\pi \longrightarrow \mathbb{N}$ is the Kostant partition function defined by $\tau(\mathbb{Z}\pi \backslash \mathbb{N}\pi) = 0$ and

$$\tau(\nu) = \# \left\{ \{k_\alpha\}_{\alpha \in \Delta^+} | \ k_\alpha \in \mathbb{N} \text{ and } \sum_{\alpha \in \Delta^+} k_\alpha \alpha = \nu \right\}, \quad \forall \nu \in \mathbb{N}\pi.$$

4.3. Take $\lambda \in \mathfrak{h}^*$. The theorem above allows us to describe the weights of the largest proper submodule N of the Verma module $\widetilde{M}(\lambda)$:

$$\Omega(N) = \bigcup_{\alpha \in \Delta_\lambda} \{\lambda - \langle \lambda + \rho, \alpha \rangle \alpha - \mathbb{N}\pi\}, \text{ where}$$
$$\Delta_\lambda = \{\alpha \in \overline{\Delta_0^+} | \ \langle \lambda + \rho, \alpha \rangle \in \mathbb{N}^+\} \cup \{\beta \in \Delta_1^+ | \ \langle \lambda + \rho, \beta \rangle \in 2\mathbb{N} + 1\}.$$

In particular, it implies the following criterion of simplicity for $\widetilde{M}(\lambda)$:

4.4. **Corollary.** *The Verma module* $\widetilde{M}(\lambda):\ \lambda\in\mathfrak{h}^*$ *is simple if and only if :*

$$\begin{array}{ll}\langle\lambda+\rho,\alpha\rangle\not\in\mathbb{N}^+, & \forall\alpha\in\overline{\Delta_0^+}\\ \langle\lambda+\rho,\alpha\rangle\not\in(2\mathbb{N}+1), & \forall\alpha\in\Delta_1^+.\end{array}$$

5. Hesselink Formula

The goal of this section is to give a formula describing a multiplicity of every simple finite dimensional module $\tilde{V}(\lambda)$ in the graded component H^n where $H:=gr_{\mathcal{F}}\mathcal{H}$. More precisely, let $\mathcal{S}^n(\mathfrak{g})\subseteq\mathcal{S}(\mathfrak{g})$ be the subspace of homogeneous elements of degree n. For a finite dimensional module $\tilde{V}(\lambda)$ we will give an explicit formula for a Poincaré series $P_\lambda(q)$ of $H=gr_{\mathcal{F}}\mathcal{H}$ which is defined by

$$P_\lambda(q):=\sum_{n=0}^{\infty}[H^n:\tilde{V}(\lambda)]q^n,\ \text{where}\ H^n:=\mathcal{S}^n(\mathfrak{g})\cap H.$$

5.1. **Notation.** Consider the group ring $\mathbb{Z}[\mathfrak{h}^*]$. Its elements are finite sums $\sum_{\lambda\in\mathfrak{h}^*}c_\lambda e^\lambda,\ c_\lambda\in\mathbb{Z}$. Our characters lie in the power series ring $\mathbb{Z}[\mathfrak{h}^*][[q]]$. For each $\mu\in\mathfrak{h}^*$ define the $\mathbb{Z}[[q]]$-linear homomorphism π_μ by

$$\pi_\mu:\mathbb{Z}[\mathfrak{h}^*][[q]]\ \to\ \mathbb{Z}[[q]],\qquad e^\lambda\mapsto\delta_{\lambda,\mu},\ \forall\lambda\in\mathfrak{h}^*.$$

5.1.1. For each $w\in W$ define the automorphism of the ring $\mathbb{Z}[\mathfrak{h}^*][[q]]$ by

$$w(e^\lambda):=e^{w\lambda},\ w(q):=q.$$

Let J be the $\mathbb{Z}[[q]]$-linear endomorphism of $\mathbb{Z}[\mathfrak{h}^*][[q]]$ given by the formula

$$J:=\sum_{w\in W}\operatorname{sgn}(w)w.$$

We use the following properties of the operator J:

(i) $J(wa)=\operatorname{sgn}(w)J(a)$ for all $w\in W$. Consequently, if there exists w such that $wa=a$ and $\operatorname{sgn}(w)=-1$ then $J(a)=0$.

(ii) Since for any $\lambda\in P^+(\pi)$ the stabilizer $\operatorname{Stab}_W\lambda$ is generated by the simple reflections it contains ([J2], A.1.1) and $P(\pi)=WP^+(\pi)$, one can conclude from (i) that

$$\operatorname{Stab}_W\mu\neq\{\mathrm{id}\}\ \Longrightarrow J(e^\mu)=0$$

for any $\mu\in P(\pi)$.

(iii) Fix a subgroup K of W and representatives $g_1,\dots,g_r$ of the left cosets W/K. Set
$$J^K:=\sum_{w\in K}\operatorname{sgn}(w)w,\qquad J^{W/K}:=\sum_{i=1}^{r}\operatorname{sgn}(g_i)g_i.$$
Then $J^{W/K}J^K=J$.

(iv) Assume that $wa=a$ for all $w\in K$. Then $J^K(ab)=aJ^K(b)$.

5.1.2. For a graded $\mathfrak{h}$-submodule N of $\mathcal{S}(\mathfrak{g})$ its graded character $\operatorname{ch}_q N$ is defined to be the element of $\mathbb{Z}[\mathfrak{h}^*][[q]]$ given by the formula
$$\operatorname{ch}_q N:=\sum_{r=0}^{\infty}\operatorname{ch}(N\cap\mathcal{S}^r(\mathfrak{g}))q^r.$$
The following relations hold
$$\begin{aligned}
\operatorname{ch}_q\Lambda\mathfrak{g}_1 &= \sum_{r=0}^{2l}(\operatorname{ch}\Lambda^r\mathfrak{g}_1)q^r = \prod_{\beta\in\Delta_1}(1+qe^{\beta})\\
\operatorname{ch}_q\mathcal{S}(\mathfrak{n}_0^{\pm}) &= \prod_{\alpha\in\Delta_0^+}(1-qe^{\pm\alpha})^{-1},\\
\operatorname{ch}_q\mathcal{S}(\mathfrak{n}^{\pm}) &= \prod_{\alpha\in\Delta_0^+}(1-qe^{\pm\alpha})^{-1}\prod_{\beta\in\Delta_1^+}(1+qe^{\pm\beta}),\\
\operatorname{ch}_q\mathcal{S}(\mathfrak{n}^+\oplus\mathfrak{n}^-) &= \operatorname{ch}_q\mathcal{S}(\mathfrak{n}_0^+\oplus\mathfrak{n}_0^-)\cdot\operatorname{ch}_q\Lambda\mathfrak{g}_1.
\end{aligned}$$

Moreover, the separation theorem implies that
$$\operatorname{ch}_q\mathcal{S}(\mathfrak{g})=\operatorname{ch}_q\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}\operatorname{ch}_q H.$$
By [Mu1], 1.2 $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}\cong\mathcal{S}(\mathfrak{h})^W$. By [J1], 8.7
$$\operatorname{ch}_q\mathcal{S}(\mathfrak{h})=\operatorname{ch}_q\mathcal{S}(\mathfrak{h})^W\sum_{w\in W}q^{l(w)}$$
so
$$\operatorname{ch}_q\mathcal{S}(\mathfrak{g})=\operatorname{ch}_q\mathcal{S}(\mathfrak{n}^-\oplus\mathfrak{n}^+)\operatorname{ch}_q\mathcal{S}(\mathfrak{h})=\operatorname{ch}_q\mathcal{S}(\mathfrak{n}^-\oplus\mathfrak{n}^+)\operatorname{ch}_q\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}\sum_{w\in W}q^{l(w)}.$$
Hence
$$\operatorname{ch}_q H=\operatorname{ch}_q\mathcal{S}(\mathfrak{n}^-\oplus\mathfrak{n}^+)\sum_{w\in W}q^{l(w)}=\prod_{\alpha\in\Delta_0}(1-qe^{\alpha})^{-1}\operatorname{ch}_q\Lambda\mathfrak{g}_1\sum_{w\in W}q^{l(w)}.\tag{8}$$

5.2. **Classical Case.** We use the previous notations for the corresponding classical objects. For the classical case the Hesselink formula (see [J1], 9.10) states that $P_\lambda(q)$ is equal to the coefficient of $e^0=1$ in the expression
$$Q_\lambda(q):=e^{-\rho}J(e^{\lambda+\rho})\prod_{\alpha\in\Delta^+}(1-qe^{-\alpha})^{-1}.\tag{9}$$
The proof (see [J1], 8.6) is essentially based on the equality
$$J(e^{\rho}\prod_{\alpha\in\Delta^+}(1-qe^{-\alpha}))=J(e^{\rho})\sum_{w\in W}q^{l(w)}.\tag{10}$$

5.3. A key point of the proof of Theorem 5.4 is the following analogue of equality 10.
Proposition.

$$J\left(e^{\rho}\Big(\prod_{\alpha\in\Delta_0^+}(1-qe^{-\alpha})\Big)(1+q^{2l}e^{\beta_1})\prod_{\beta\in\Delta_1^+\setminus\{\beta_1\}}(1+e^{\beta})\right)=J(e^{\rho})\operatorname{ch}_q\Lambda\mathfrak{g}_1\sum_{w\in W}q^{l(w)} \tag{11}$$

where β_1 is the maximal odd root.

Proof. The proposition will be proven in 5.3.1—5.3.3. First of all we need the following technical lemma.

5.3.1. **Lemma.** *Set $\Gamma_\bullet:=\{\sum_{i=2}^{l}k_i\beta_i,\quad k_i\in\{0,1\}\}$. For any $r=0,1,\ldots,2l-1$ there exists a unique $\gamma_r\in\Gamma_\bullet$ such that $\mathrm{Stab}_W(\rho-r\beta_1+\gamma_r)=\{\mathrm{id}\}$. Moreover $\rho-r\beta_1+\gamma_r=w_r\rho$ for some $w_r\in W$ and $\mathrm{sgn}(w_r)=(-1)^r$.*

Proof. Recall that the set $\{\beta_i\}_{i=1}^l$ is an orthonormal basis of $\mathfrak{h}^*$ and W acts on this basis by signed permutations. Therefore the stabilizer of $\lambda\in\mathbb{Q}\pi$ is trivial iff $|(\lambda,\beta_i)|$ are pairwise distinct non-zero values.

Assume that $\mathrm{Stab}_W(\rho-r\beta_1+\gamma)=\{\mathrm{id}\}$ for some $\gamma\in\Gamma_\bullet$ and set $\lambda:=\rho-r\beta_1+\gamma$. Recall that $(\rho,\beta_i)=l+1/2-i$. Since $0\leq r<2l$ one has

$$\{|(\lambda,\beta_i)|\}_{i=1}^l\subseteq\{l+1/2-i\}_{i=1}^l=\{|(\rho,\beta_i)|\}_{i=1}^l.$$

Taking into account that all $|(\lambda,\beta_i)|$ are pairwise distinct we conclude that $\{|(\lambda,\beta_i)|\}_{i=1}^l=\{|(\rho,\beta_i)|\}_{i=1}^l$. This implies that $\lambda=w\rho$ for some $w\in W$ and that γ is determined by the value of $(\lambda,\beta_1)=l-1/2-r$. The explicit expressions of γ and w for a fixed r are:

$$\begin{cases} r=0 & \gamma=0, & w=id\\ 1\leq r<l & \gamma=\sum_{i=2}^{r+1}\beta_i, & w=s_1\ldots s_r\\ l\leq r<2l & \gamma=\sum_{i=2}^{2l-r}\beta_i, & w=s_1\ldots s_ls_{l-1}\ldots s_{2l-r}\end{cases}$$

This completes the proof of the lemma. ∎

5.3.2. Let $\Delta_\bullet$ be the subsystem of Δ_0 generated by $\pi\setminus\{\beta_1-\beta_2\}$ and $W_\bullet$ be the corresponding Weyl group that is the subgroup of W generated by the reflections $s_{\alpha_2},\ldots,s_{\alpha_l}$. Set $\Delta_\bullet^+:=\Delta_\bullet\cap\Delta^+$, $\rho_\bullet:=\sum_{\alpha\in\Delta_\bullet^+}\alpha/2$. Observe that $\Delta_\bullet$ is the root system corresponding to the simple Lie algebra $\mathrm{sp}(l-1)$ and that $\rho_\bullet=\rho_0-l\beta_1$. By [H], 3.15 for the Weyl group

W_n of the Lie algebra sp(n) one has $\sum_{w\in W_n} q^{l(w)} = (1-q)^{-n}\prod_{i=1}^{n}(1-q^{2i})$ and thus

$$(1-q^{2l})\sum_{w\in W_\bullet} q^{l(w)} = (1-q)\sum_{w\in W} q^{l(w)} \tag{12}$$

5.3.3. Observe that

$$\begin{aligned}(1-q)\prod_{\alpha\in\Delta_0^+\setminus\Delta_\bullet^+}(1-qe^{-\alpha}) &= (1-qe^{-2\beta_1})(1-q)\prod_{i=2}^{l}(1-qe^{-\beta_1-\beta_i})(1-qe^{-\beta_1+\beta_i})\\ &= \prod_{i=1}^{l}(1-qe^{-\beta_1-\beta_i})(1-qe^{-\beta_1+\beta_i})\\ &= \sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)\cdot(-q)^r e^{-r\beta_1}.\end{aligned} \tag{13}$$

The characters $\mathrm{ch}(\Lambda^r\mathfrak{g}_1)$ are stable under the action of W since $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. By the property (iv) of 5.1.1 with $K=W$, one has

$$\begin{aligned}&(1-q)J\left(e^{\rho}\Big(\prod_{\alpha\in\Delta_0^+}(1-qe^{-\alpha})\Big)(1+q^{2l}e^{\beta_1})\prod_{\beta\in\Delta_1^+\setminus\{\beta_1\}}(1+e^{\beta})\right)=\\ &\sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)(-q)^rJ\left(e^{\rho-r\beta_1}(1+q^{2l}e^{\beta_1})\prod_{\alpha\in\Delta_\bullet^+}(1-qe^{-\alpha})\prod_{i=2}^{l}(1+e^{\beta_i})\right).\end{aligned} \tag{14}$$

Decompose $J=J^{W/W_\bullet}J^{W_\bullet}$ (see (iii) of 5.1.1) and observe that the elements

$$e^{\beta_1}\ \text{and}\ e^{\rho-\rho_\bullet}\prod_{i=2}^{l}(1+e^{\beta_i}) = e^{l\beta_1}\prod_{i=2}^{l}(e^{-\beta_i/2}+e^{\beta_i/2})$$

are stable under the action of $W_\bullet$. Using the formula (10) with respect to $W_\bullet$ one can rewrite the right hand side of (14) as

$$\begin{aligned}&\sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)(-q)^rJ^{W/W_\bullet}\left(e^{\rho-\rho_\bullet-r\beta_1}(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})J^{W_\bullet}\Big(e^{\rho_\bullet}\prod_{\alpha\in\Delta_\bullet^+}(1-qe^{-\alpha})\Big)\right)=\\ &\sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)(-q)^rJ^{W/W_\bullet}\left(e^{\rho-\rho_\bullet-r\beta_1}(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})J^{W_\bullet}(e^{\rho_\bullet})\sum_{w\in W_\bullet}q^{l(w)}\right)=\\ &\Big(\sum_{w\in W_\bullet}q^{l(w)}\Big)\sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)(-q)^rJ\left(e^{\rho-r\beta_1}(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\right)=\\ &\Big(\sum_{w\in W_\bullet}q^{l(w)}\Big)\sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)a_r,\end{aligned} \tag{15}$$

where $a_r=(-q)^rJ\Big(e^{\rho-r\beta_1}(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\Big)$.

One can simplify the expression obtained in the following way. For $r \in \{1, \dots, 2l-1\}$ take w_r, w_{r-1} as in Lemma 5.3.1. Then

$$a_r = (-q)^r J(e^{w_r\rho} + q^{2l} e^{w_{r-1}\rho}) = (1-q^{2l})q^r J(e^\rho). \tag{16}$$

For $r = 0, 2l$ we have $\mathrm{ch}(\Lambda^0 \mathfrak{g}_1) = \mathrm{ch}(\Lambda^{2l}\mathfrak{g}_1) = 1$, therefore

$$a_0 + a_{2l} = J\left((e^\rho + q^{2l}e^{\rho-2l\beta_1})(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\right).$$

Consider the reflection $s = s_{\beta_1} \in W$. One has $s\rho = \rho - (2l-1)\beta_1$. Thus the expression $(e^{\rho+\beta_1} + e^{\rho-2l\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})$ is stable under the action of s. Since $\mathrm{sgn}(s) = -1$, property (i) of 5.1.1 implies that

$$J\left((e^{\rho+\beta_1} + e^{\rho-2l\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\right) = 0. \tag{17}$$

Using Lemma 5.3.1 with respect to $r = 0$ and $r = 2l-1$, we obtain

$$J\left((e^\rho + q^{4l}e^{\rho-(2l-1)\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\right) = J(e^\rho - q^{4l}e^\rho) = J(e^\rho)(1-q^{2l})(1+q^{2l}). \tag{18}$$

Summarizing (17) and (18) we get

$$a_0 + a_{2l} = J\left((e^\rho + q^{2l}e^{\rho-2l\beta_1})(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\right) = J(e^\rho)(1-q^{2l})(1+q^{2l}). \tag{19}$$

By the substitution of (16) and (19) into (15) one can rewrite (14) as

$$\begin{aligned}
&(1-q)J\left(e^\rho\Bigl(\prod_{\alpha\in\Delta_0^+}(1-qe^{-\alpha})\Bigr)(1+q^{2l}e^{\beta_1})\prod_{\beta\in\Delta_1^+\setminus\{\beta_1\}}(1+e^\beta)\right) = \\
&\sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)q^r J(e^\rho)(1-q^{2l})\left(\sum_{w\in W_\bullet} q^{l(w)}\right) = \\
&\sum_{r=0}^{2l}\mathrm{ch}(\Lambda^r\mathfrak{g}_1)q^r J(e^\rho)(1-q)\left(\sum_{w\in W} q^{l(w)}\right) \quad \text{by (12)}
\end{aligned} \tag{20}$$

This completes the proof of Proposition 5.3. □

The following theorem is the analogue of Hesselink formula for $\mathrm{osp}(1,2l)$.

5.4. **Theorem.** *For any $\lambda \in P^+(\pi)$, the Poincaré series $P_\lambda(q)$ is equal to the coefficient of $e^0 = 1$ in the expression*

$$Q_\lambda(q) := e^{-\rho} J(e^{\lambda+\rho}) \Big(\prod_{\alpha\in\Delta_0^+} (1-qe^{-\alpha})^{-1} \Big) (1+q^{2l}e^{-\beta_1}) \prod_{\beta\in\Delta_1^+\setminus\{\beta_1\}} (1+e^{-\beta}) \tag{21}$$

where β_1 is the maximal odd root.

Proof. Observe that $\{\operatorname{ch}\widetilde{V}(\lambda) : \lambda\in P^+(\pi)\}$ are linearly independent. Hence in order to find the Poincaré series $P_\lambda(q)$ one can decompose the graded character $\operatorname{ch}_q H$ in terms of $\{\operatorname{ch}\widetilde{V}(\lambda) : \lambda\in P^+(\pi)\}$.

Recall the character formula of $\widetilde{V}(\lambda),\ \lambda\in P^+(\pi)$, computed by Kac in [K1], (3):

$$\operatorname{ch}\widetilde{V}(\lambda) = e^{-\rho}J(e^{\lambda+\rho}) \prod_{\alpha\in\Delta_0^+}(1-e^{-\alpha})^{-1} \prod_{\beta\in\Delta_1^+}(1+e^{-\beta}). \tag{22}$$

Therefore

$$J(e^\rho)\operatorname{ch}_q H = J(e^\rho)\sum_{\lambda\in P^+(\pi)} P_\lambda(q)\operatorname{ch}\widetilde{V}(\lambda) = \sum_{\lambda\in P^+(\pi)} P_\lambda(q) J(e^{\lambda+\rho}),\ \text{so}$$

$$P_\lambda(q) = \pi_{\lambda+\rho}\Big(J(e^\rho)\operatorname{ch}_q H\Big). \tag{23}$$

On the other hand,

$$\begin{aligned} J(e^\rho)\operatorname{ch}_q H &= J(e^\rho)\textstyle\prod_{\alpha\in\Delta_0}(1-qe^\alpha)^{-1}\operatorname{ch}_q \Lambda\mathfrak{g}_1 \sum_{w\in W} q^{l(w)} \text{ by (8)} \\ &= \prod_{\alpha\in\Delta_0}(1-qe^\alpha)^{-1} J\left(e^\rho\Big(\prod_{\alpha\in\Delta_0^+}(1-qe^{-\alpha})\Big)(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\right) \\ &\qquad \text{by Proposition 5.3} \\ &= J\left(e^\rho\Big(\prod_{\alpha\in\Delta_0^+}(1-qe^\alpha)^{-1}\Big)(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\right) \\ &\qquad \text{by property (iv) of 5.1.1 with } K=W. \end{aligned} \tag{24}$$

Define the $\mathbb{Z}[[q]]$-automorphism ι of the ring $\mathbb{Z}[\mathfrak{h}^*][[q]]$ by

$$\iota : e^\lambda \mapsto e^{-\lambda},\ \forall\lambda\in\mathfrak{h}^*.$$

Clearly $\pi_0\circ\iota = \pi_0$. For any $a\in\mathbb{Z}[\mathfrak{h}^*][[q]]$ one has

$$\begin{aligned} \pi_\mu J(a) &= \sum_{w\in W}\operatorname{sgn}(w)\pi_{w\mu}(a) = \sum_{w\in W}\operatorname{sgn}(w)\pi_0(e^{-w\mu}a) = \sum_{w\in W}\operatorname{sgn}(w)\pi_0\circ\iota(e^{-w\mu}a) \\ &= \pi_0\left(\sum_{w\in W}\operatorname{sgn}(w)(e^{w\mu}\iota(a))\right) = \pi_0\left(J(e^\mu)\iota(a)\right). \end{aligned} \tag{25}$$

It was already shown (see (24)) that

$$P_\lambda(q)=\pi_{\lambda+\rho}J\Big(e^\rho\prod_{\alpha\in\Delta_0^+}(1-qe^\alpha)^{-1}(1+q^{2l}e^{\beta_1})\prod_{i=2}^{l}(1+e^{\beta_i})\Big).$$

Using (25) we get

$$P_\lambda(q)=\pi_0\Big(J(e^{\lambda+\rho})e^{-\rho}\prod_{\alpha\in\Delta_0^+}(1-qe^{-\alpha})^{-1}(1+q^{2l}e^{-\beta_1})\prod_{i=2}^{l}(1+e^{-\beta_i})\Big).$$

This establishes the theorem. □

5.4.1. *Remark.* Observe that $Q_\lambda(1)=\operatorname{ch}\widetilde{V}(\lambda)$ (see (22)). Therefore $P_\lambda(1)$ is equal to the coefficient of $e^0=1$ in $\operatorname{ch}\widetilde{V}(\lambda)$ that is $\dim\widetilde{V}(\lambda)_0$. On the other hand, one has

$$P_\lambda(1)=\sum_{n=0}^{\infty}[H^n:\widetilde{V}(\lambda)]=[H:\widetilde{V}(\lambda)]=[\mathcal{H}:\widetilde{V}(\lambda)]$$

that again gives the second assertion of the separation theorem $[\mathcal{H}:\widetilde{V}(\lambda)]=\dim\widetilde{V}(\lambda)_0$.

5.4.2. *Remark.* One has

$$\pi_{\lambda+\rho}J(a)=\sum_{w\in W}\operatorname{sgn}(w)\pi_{w(\lambda+\rho)}(a)=\sum_{w\in W}\operatorname{sgn}(w)\pi_{w.\lambda}(e^{-\rho}a)$$

where $w.\lambda=w(\lambda+\rho)-\rho$ is a twisted action of W on $\mathfrak{h}^*$. Thus one can reformulate Theorem 5.4 as follows: for all $\lambda\in P^+(\pi),\ n\in\mathbb{N}$

$$[H^n:\widetilde{V}(\lambda)]=\sum_{w\in W}(-1)^{l(w)}P_n(w.\lambda),$$

where coefficients $P_n(\mu)$ are zero for $\mu\in P(\pi)\setminus\mathbb{N}\pi$ and for $\mu\in\mathbb{N}\pi$ are given by the formula

$$\Big(\prod_{\alpha\in\Delta_0^+}(1-qe^\alpha)^{-1}\Big)(1+q^{2l}e^{\beta_1})\prod_{\beta\in\Delta_1^+\setminus\{\beta_1\}}(1+e^\beta)=\sum_{r=0}^{\infty}\sum_{\nu\in\mathbb{N}\pi}P_r(\nu)e^\nu q^r.$$

6. The Parthasarathy–Ranga-Rao–Varadarajan determinants

The main result of this Section– Theorem 6.5– calculates the factorization of the Parthasarathy–Ranga-Rao–Varadarajan (PRV) determinants. In 6.1, 6.2 we give a construction of these determinants, in 6.3 we study in details the case $l=1$, and in 6.4 we compute a bound for the total degrees of the PRV determinants. The factorization formula for the PRV determinants is given in 6.5. Subsections 6.6– 6.11 are devoted to the proof of this formula.

6.1. In order to introduce the Parthasarathy–Ranga-Rao–Varadarajan (PRV) determinants, start with a little lemma. Retain notation of 4.1. Observe that $\Upsilon(\mathcal{U}(\mathfrak{g})_\mu) = 0$ for all $\mu \neq 0$. In particular, $\Upsilon(M) = \Upsilon(M_0)$ for any $ad\,\mathfrak{g}$-submodule M of $\mathcal{U}(\mathfrak{g})$.

6.1.1. **Lemma.** *Let M be an $\mathrm{ad}\,\mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})$, and $\widetilde{V}(\lambda)$ a simple $\mathfrak{g}$-module of highest weight λ, then $M \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{V}(\lambda) \Longleftrightarrow \Upsilon(M_0)(\lambda) = 0$.*

Proof. The proof is the same as that of Lemma 7.2 in [J1]. However, we present it here since this is a very important (and quite easy) fact. Let v_λ be a highest weight vector of $\widetilde{V}(\lambda)$ and $\widetilde{V}(\lambda)_- := \mathfrak{n}^- \widetilde{V}(\lambda)$. One has

$$\Upsilon(M)(\lambda) = 0 \Longleftrightarrow Mv_\lambda \subset \widetilde{V}(\lambda)_-.$$

On the other hand, from the $ad\,\mathfrak{g}$-invariance of M one has $\mathcal{U}(\mathfrak{n}^-)M = M\mathcal{U}(\mathfrak{n}^-)$ and so the assumption $\Upsilon(M_0)(\lambda) = 0$ implies

$$M\widetilde{V}(\lambda) = M\mathcal{U}(\mathfrak{n}^-)v_\lambda = \mathcal{U}(\mathfrak{n}^-)Mv_\lambda \subset \mathcal{U}(\mathfrak{n}^-)\widetilde{V}(\lambda)_- \subset \widetilde{V}(\lambda)_- \underset{\neq}{\subset} \widetilde{V}(\lambda).$$

The $ad\,\mathfrak{g}$-invariance of M implies that $\mathcal{U}(\mathfrak{g})M = M\mathcal{U}(\mathfrak{g})$ and consequently $M\widetilde{V}(\lambda)$ is a submodule of $\widetilde{V}(\lambda)$. Hence $M\widetilde{V}(\lambda) = 0$. $\square$

6.2. **The construction of the PRV determinants.** The separation theorem leads to the following contruction. Fix a simple finite dimensional module $\widetilde{V}(\lambda)$ of the highest weight $\lambda \in P^+(\pi)$ and let $C(\lambda)$ be the corresponding isotypical component in $\mathcal{H}$.

$$C(\lambda) = \bigoplus_{j=1}^{\dim V(\lambda)_0} V^{(j)}, \quad \widetilde{V}(\lambda) \xrightarrow{\sim} V^{(j)}. \tag{26}$$

Choose a basis $\{v_i\}$ of $\widetilde{V}(\lambda)_0$ and let $\{v_i^j\}$ be the corresponding basis of $V_0^{(j)}$. Consider the matrix $PRV^\lambda := (\Upsilon(v_i^j))$. Lemma 6.1.1 implies that the j^{th} column of the matrix PRV^λ is zero at a point $\mu \in \mathfrak{h}^*$ iff $V^{(j)} \subset \mathrm{Ann}\,\widetilde{V}(\mu)$. Notice that the matrix PRV^λ depends on the choice of the decomposition of the isotypical component $C(\lambda)$, on the choice of the basis of $\widetilde{V}(\lambda)_0$ and on the choice of the isomorphisms from each $V^{(j)}$ to $\widetilde{V}(\lambda)$. For any change of these parameters, the new matrix one obtains is of the form $M(PRV^\lambda)N$ where M, N are invertible square complex matrices. Therefore, the corank of the PRV^λ is correctly defined and $\det PRV^\lambda \in \mathcal{S}(\mathfrak{h})$ is defined up to a non-zero scalar. Hence, for any $\mu \in \mathfrak{h}^*$, one has the equivalence:

$$\det PRV^\lambda(\mu) = 0 \Longleftrightarrow C(\lambda) \cap \mathrm{Ann}_{\mathcal{H}}\,\widetilde{V}(\mu) \neq 0.$$

Moreover

6.2.1. **Lemma.** *Take $\lambda \in P^+(\pi)$ and $\mu \in \mathfrak{h}^*$. Then*

(i) $\operatorname{corank} PRV^\lambda(\mu) = [\operatorname{Ann}_{\mathcal{H}} \widetilde{V}(\mu) : \widetilde{V}(\lambda)]$
(ii) *μ is a zero of $\det PRV^\lambda$ of order $\geq [\operatorname{Ann}_{\mathcal{H}} \widetilde{V}(\mu) : \widetilde{V}(\lambda)]$*
(iii) *for any $\lambda \in P^+(\pi)$, the polynomial $\det PRV^\lambda$ is not identically zero.*

Proof. From the comments above (i) and (ii) are straightforward. The proof of (iii) coincides with the proof in 9.11.4, [J1]. $\square$

6.3. **Case $l = 1$.** This case was completely described by Pinczon in [Pi]. In this case $\pi = \Delta_1^+ = \{\beta\},\ \Delta_0^+ = \{2\beta\},\ \rho = \frac{1}{2}\beta,\ \mathfrak{h}^* = \mathbb{C}\beta, \mathfrak{h} = \mathbb{C}\varphi(\beta),\ P^+(\pi) = \mathbb{N}\beta$.

6.3.1. Denote by $\widetilde{V}(n)$ a simple module of the highest weight $n\beta$. Call a simple module V *odd* if $V \cong \widetilde{V}(n)$ for some odd number n.

Since $\dim \widetilde{V}(m)_0 = 1$ for all $m \in \mathbb{N}$ the separation theorem implies that

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} \widetilde{V}(m).$$

Taking into account that for any $m \in \mathbb{N},\ s_\beta.(m\beta) = -(m+1)\beta \notin P^+(\pi)$, we conclude from Remark 5.4.2 that

$$[H^n : \widetilde{V}(m)] = P_n(m\beta)$$

where the coefficients $P_n(m\beta)$ are given by the formula

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(m\beta)e^{m\beta}q^n = (1-qe^{2\beta})^{-1}(1+q^2e^{\beta}) = 1 + qe^{2\beta} + \sum_{n=2}^{\infty} q^n(e^{2n\beta} + e^{(2n-3)\beta}).$$

Hence

$$H^n = \widetilde{V}(2n) \oplus \widetilde{V}(2n-3) \text{ for } n \geq 2 \tag{27}$$

Since $\mathcal{S}(\mathfrak{g}) = H \otimes \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, we conclude that for $m = 2n, 2n-3$ one has $[\mathcal{S}^k(\mathfrak{g}) : \widetilde{V}(m)] = 0$ for $0 \leq k < n$ and $[\mathcal{S}^n(\mathfrak{g}) : \widetilde{V}(m)] = 1$. Hence $\mathcal{F}^n$ contains a unique copy of $\widetilde{V}(2n)$ and $\widetilde{V}(2n-3)$ which lie in $\mathcal{H}$. This shows in particular that, for any $n \geq 0$, the degrees of $\det PRV^{2n\beta}$ and $\det PRV^{2n+1}$ are less than or equal to n and $n+2$ respectively.

6.3.2. Let V be the copy of $\widetilde{V}(2n)$ in $\mathcal{H}$ and v its highest weight vector. For any $k \geq 0$, $V \subset \operatorname{Ann}_{\mathcal{H}} \widetilde{V}(k)$ iff $v\widetilde{V}(k) = 0$. As $\Omega(\widetilde{V}(k)) = \{k, k-1, \ldots, 0, \ldots, -k+1, -k\}$, the condition $n > k$ is sufficient to ensure that $v\widetilde{V}(k) = 0$ and therefore that $V \subset \operatorname{Ann}_{\mathcal{H}} \widetilde{V}(k)$. So $\det PRV^{2n\beta}$ is divisible by $\prod_{k=0}^{n-1}(\varphi(\beta) - k)$. Analogously we obtain also the divisibility of $\det PRV^{(2n+1)\beta}$ by $\prod_{k=0}^{n}(\varphi(\beta) - k)$. Taking into account that the degree of $\det PRV^{2n\beta} \leq n$ we conclude that, up to a non-zero scalar,

$$\det PRV^{2n\beta} = \prod_{k=0}^{n-1}(\varphi(\beta) - k).$$

6.3.3. The factorization of $\det PRV^{(2n+1)\beta}$ requires some extra work. The Casimir operator C_0 of $\mathcal{U}(\mathfrak{g}_0)$ acts on a $\mathfrak{g}_0$-Verma module $M(\mu)$ of highest weight μ by the scalar $(\mu, \mu+2\rho_0)$. Since $\widetilde{M}(\mu) = M(\mu) \oplus M(\mu-\beta)$ as a $\mathfrak{g}_0$-module, it follows that C_0 acts on $\widetilde{M}(\mu)$ by a scalar iff $\mu = -\beta/2 = -\rho$. Hence $(C_0 + \frac{3}{4}) \in \operatorname{Ann} \widetilde{M}(-\rho)$. Therefore $\operatorname{Ann} \widetilde{M}(-\rho)$ contains $V := ad\mathcal{U}(\mathfrak{g})(C_0 + \frac{3}{4})$. Since $\mathfrak{g}_0$ acts trivially on $(C_0 + \frac{3}{4})$ and $\Omega(\mathfrak{g}_1) = \{\beta, -\beta\}$, it follows that $V \subseteq V^{(1)} \oplus V^{(2)}$ where $V^{(1)} \cong \widetilde{V}(1)$, $V^{(2)} \cong \widetilde{V}(0)$. On the other hand, $C_0 \notin \mathcal{Z}(\mathfrak{g})$ so $V^{(2)} \neq V$. Hence $V^{(1)} \subseteq V$. Since $C_0 \in \mathcal{F}^2$, we conclude that $\operatorname{Ann} \widetilde{M}(-\rho)$ contains a copy $V^{(1)}$ of $\widetilde{V}(1)$ and this copy lies in $\mathcal{F}^2$. Thus $V^{(1)}$ lies in $\mathcal{H}$. Fix a primitive vector a of $V^{(1)}$. Taking into account that $X_{2\beta}$ is a primitive vector of $\mathfrak{g} \cong \widetilde{V}(2)$ and $\mathcal{U}(\mathfrak{g})$ is a domain, we conclude that the product $X_{2\beta}^k a$ is also a primitive vector of the weight $(2k+1)\beta$ and so $ad\mathcal{U}(\mathfrak{g})(X_{2\beta}^k a) \cong \widetilde{V}(2k+1)$. Since $X_{2\beta}^k a \in \mathcal{F}^{k+2}$ and $a \in \operatorname{Ann} \widetilde{M}(-\rho)$, it follows that $ad\mathcal{U}(\mathfrak{g})(X_{2\beta}^k a) \subset \operatorname{Ann}_{\mathcal{H}} \widetilde{M}(-\rho)$. Hence

$$\operatorname{Ann}_{\mathcal{H}} \widetilde{M}(-\rho) \supseteq \bigoplus_{k=0}^{\infty} \widetilde{V}(2k+1). \tag{28}$$

Since $W.(-\rho) = \{-\rho\}$, the module $\widetilde{M}(-\rho)$ is simple. So (28) means that $\varphi(\beta) + \frac{1}{2}$ divides all $\det PRV^{(2n+1)\beta}$, $n \geq 0$. We conclude from 6.3.2 that, up to a non-zero scalar,

$$\det PRV^{(2n+1)\beta} = (\varphi(\beta) + \frac{1}{2}) \prod_{k=0}^{n} (\varphi(\beta) - k).$$

From the factorization of $\det PRV^{2n\beta}$ and $\det PRV^{(2n+1)\beta}$ we derive that (28) is an equality and that $\operatorname{Ann}_{\mathcal{H}} \widetilde{M}(\mu) = 0$ if $\widetilde{M}(\mu)$ simple and $\mu \neq -\rho$. Since $W.(-\rho) = \{-\rho\}$, $\operatorname{Hom}_{\mathfrak{g}}(\widetilde{M}(-\rho), \widetilde{M}(\mu)) = 0$ for any $\mu \neq -\rho$. Hence $\operatorname{Ann}_{\mathcal{H}} \widetilde{M}(\mu) = 0$ for any $\mu \neq -\rho$. We summarize:

$$\begin{aligned} \mu \neq -\rho &\implies \operatorname{Ann}_{\mathcal{H}} \widetilde{M}(\mu) = 0 \\ \mu = -\rho &\implies \operatorname{Ann}_{\mathcal{H}} \widetilde{M}(-\rho) = \bigoplus_{i=0}^{\infty} \widetilde{V}(2i+1) \end{aligned}$$

In the sequel we will need the following

6.3.4. **Lemma.** *Take $\mathfrak{g} = \operatorname{osp}(1,2)$. For any odd submodule V of $\mathcal{U}(\mathfrak{g})$ (see 6.3.1) and any $v \in V$, the Harish-Chandra projection $\Upsilon(v)$ is divisible by $(\varphi(\beta) + \frac{1}{2})$.*

Proof. Since $\mathcal{Z}(\mathfrak{g})$ acts on $\widetilde{M}(-\rho)$ by scalars, the separation theorem implies that as $\mathfrak{g}$-module

$$\mathcal{U}(\mathfrak{g})/(\operatorname{Ann} \widetilde{M}(-\rho)) \cong \mathcal{H}/(\operatorname{Ann}_{\mathcal{H}} \widetilde{M}(-\rho)) = \bigoplus_{i=0}^{\infty} \widetilde{V}(2i).$$

Thus $\operatorname{Ann} \widetilde{M}(-\rho)$ contains all odd submodules of $\mathcal{U}(\mathfrak{g})$. By Lemma 6.1.1, $\Upsilon(v)(-\rho) = 0$ for all $v \in V$. Since $\Upsilon(v)$ is just a polynomial of one variable, it follows that $\Upsilon(v)$ is divisible by $\varphi(\beta) + (\rho, \beta) = \varphi(\beta) + \frac{1}{2}$ as required. $\square$

6.4. **A bound for the total degrees of PRV determinants.** Retain notations of 6.2. Fix a finite dimensional module $\widetilde{V}(\lambda)$ and a decomposition (26) of its isotypical component $C(\lambda)$ in $\mathcal{H}$. For each copy $V^{(i)} \subset \mathcal{H}$ let n_i be such that $gr_{\mathcal{F}} V^{(i)} \subset H^{n_i}$. The subspaces $\mathcal{F}^{(k)}$ are stable under the projection Υ so $\Upsilon(V^{(i)}) \subset \mathcal{F}^{(n_i)} \cap \mathcal{U}(\mathfrak{h})$. Thus any entry $\Upsilon(v^i_j)$ of i-th column of PRV^λ has degree (as polynomial in $S(\mathfrak{h})$) less than or equal to n_i. Using prime to denote the derivative with respect to q we obtain

$$\text{degree } \det PRV^\lambda \leq \sum_{r=0}^{\infty} r[H^r : \widetilde{V}(\lambda)] = P'_\lambda(1)$$

where $\det PRV^\lambda$ is considered as a polynomial in $S(\mathfrak{h})$.

Using notations of 5.1 and Theorem 5.4 we can rewrite (21) as $P_\lambda(q) = \pi_0(Q_\lambda(q))$. Hence

$$\text{degree } \det PRV^\lambda \leq \pi_0(Q'_\lambda(1)). \tag{29}$$

Taking the derivative of (21) on q, we obtain

$$Q'_\lambda(q) = Q_\lambda(q) \left(\sum_{\alpha \in \Delta_0^+} e^{-\alpha}(1 - qe^{-\alpha})^{-1} + 2lq^{2l-1} e^{-\beta_1}(1 + q^{2l} e^{-\beta_1})^{-1} \right).$$

By Remark 5.4.1, $Q_\lambda(1) = \operatorname{ch} \widetilde{V}(\lambda)$. Thus

$$\begin{aligned} Q'_\lambda(1) &= Q_\lambda(1) \left(\sum_{\alpha \in \Delta_0^+} e^{-\alpha}(1 - e^{-\alpha})^{-1} + 2le^{-\beta_1}(1 + e^{-\beta_1})^{-1} \right) \\ &= \sum_{\alpha \in \Delta_0^+} \sum_{m=1}^{\infty} e^{-m\alpha} \operatorname{ch} \widetilde{V}(\lambda) + 2l \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m\beta_1} \operatorname{ch} \widetilde{V}(\lambda) \end{aligned} \tag{30}$$

Since $\pi_0(e^{-\mu} \operatorname{ch} \widetilde{V}(\lambda)) = \dim \widetilde{V}(\lambda)_\mu$ for any $\mu \in P(\pi)$, the formulas (29), (30) imply

$$\text{degree } \det PRV^\lambda \leq \sum_{m=1}^{\infty} \left(\sum_{\alpha \in \Delta_0^+} \dim \widetilde{V}(\lambda)_{m\alpha} + 2l(-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta_1} \right) \tag{31}$$

6.5. **Factorization of PRV determinants.** Let us formulate the main theorem of this Section:

Theorem. *For all $\lambda \in P^+(\pi)$, one has up to a non-zero scalar,*

$$\det PRV^\lambda = \underbrace{\prod_{\alpha\in\overline{\Delta_0^+}}\prod_{m=1}^{\infty}\Big[\varphi(\alpha)+(\rho,\alpha)-\frac{1}{2}m(\alpha,\alpha)\Big]^{\dim \widetilde{V}(\lambda)_{m\alpha}}}_{standard\ factors}\times$$

$$\underbrace{\prod_{\alpha\in\Delta_1^+}\prod_{m=1}^{\infty}\Big[\varphi(\alpha)+(\rho,\alpha)-\frac{1}{2}(2m-1)(\alpha,\alpha)\Big]^{dim\widetilde{V}(\lambda)_{(2m-1)\alpha}}}_{standard\ factors}\times$$

$$\underbrace{\prod_{\alpha\in\Delta_1^+}\Big[\varphi(\alpha)+(\rho,\alpha)\Big]^{\sum_{i=1}^{\infty}(-1)^{i+1}\dim\widetilde{V}(\lambda)_{i\alpha}}}_{exotic\ factors}$$

The rest of this section is devoted to the proof of this theorem.

6.6. **Standard factors of the PRV determinants.** In the subsections 6.6— 6.7 we shall explain briefly how certain factors of the PRV determinants can be obtained following the procedure used by Joseph in the classical case [J1], Chapter 9. These factors will be called "standard factors".

Let first fix some notations. Let $M = M_0 \oplus M_1$ be a $\mathfrak{g}$-module. We endow the space $\mathrm{Hom}_{\mathbb{C}}(M,M)$ of $\mathbb{C}$-linear endomorphisms of M with the natural supervector space structure. The $\mathfrak{g}$-module structure on $\mathrm{Hom}_{\mathbb{C}}(M,M)$ is defined on homogenous elements $x\in\mathfrak{g},\ f\in \mathrm{Hom}_{\mathbb{C}}(M,M)$ by the formula

$$(xf)(m) = x(f(m)) - (-1)^{|x||f|}f(xm)$$

The locally finite part of $\mathrm{Hom}_{\mathbb{C}}(M,M)$ with respect to this $\mathfrak{g}$-module structure will be denoted $\mathrm{F}(M,M)$. For the above $\mathfrak{g}$-module structure on $\mathrm{Hom}_{\mathbb{C}}(M,M)$ the natural map $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(M,M)$ is a $\mathfrak{g}$-module map for the adjoint action on $\mathcal{U}(\mathfrak{g})$. The image is contained in $\mathrm{F}(M,M)$.

Take $\mu\in\mathfrak{h}^*$ and consider the induced morphism

$$\mathcal{H}/\operatorname{Ann}_{\mathcal{H}}\widetilde{V}(\mu)\longrightarrow \mathrm{F}(\widetilde{V}(\mu),\widetilde{V}(\mu)).$$

The idea to find the "standard zeroes" of $\det PRV^\lambda$ is to construct some μ such that the multiplicity of $\widetilde{V}(\lambda)$ in $\mathrm{F}(\widetilde{V}(\mu),\widetilde{V}(\mu))$ is less than the multplicity of $\widetilde{V}(\lambda)$ in $\mathcal{H}$.

Define for $\alpha\in\Delta_{irr}$,

$$\Lambda_{m,\alpha} := \{\mu\in\mathfrak{h}^* |\ \langle\mu+\rho,\alpha\rangle = m,\ \ \langle\mu+\rho,\beta\rangle\notin\mathbb{Z}\ ,\forall\beta\in\Delta_{irr}^+\backslash\{\alpha\}\}$$

for all $m\in\mathbb{N}^+$ if $\alpha\in\overline{\Delta_0^+}$ and for all odd m if $\alpha\in\Delta_1^+$. For such choices of m and α, the set $\Lambda_{m,\alpha}$ is obviously non-empty.

One has the

6.6.1. **Theorem.** *Let $\lambda \in P^+(\pi)$, $\alpha \in \Delta^+_{irr}$, $m \in \mathbb{N}^+$ (assumed to be odd if α is odd), $\mu \in \Lambda_{m,\alpha}$ then*

$$\left[\mathrm{F}(\widetilde{V}(\mu), \widetilde{V}(\mu)) : \widetilde{V}(\lambda)\right] = \dim \widetilde{V}(\lambda)_0 - \dim \widetilde{V}(\lambda)_{m\alpha}.$$

Assume m, α chosen as above. Then by Lemma 6.2.1 and the theorem, $\mu \in \Lambda_{m,\alpha}$ is a zero of $\det PRV^\lambda$ of order $\geq \dim \widetilde{V}(\lambda)_{m\alpha}$. The same holds for the Zariski closure of $\Lambda_{m,\alpha}$, namely for the hyperplane $(\mu+\rho,\alpha) - m\dfrac{(\alpha,\alpha)}{2}$. This means that $\left[\varphi(\alpha) + (\rho,\alpha) - m\dfrac{(\alpha,\alpha)}{2}\right]$ divides $\det PRV^\lambda$. Consequently we obtain the

6.6.2. **Corollary.** *For any $\lambda \in P^+(\pi)$, the determinant $\det PRV^\lambda$ is divisible by*

$$\prod_{\alpha\in\overline{\Delta^+_0}} \prod_{m=1}^{\infty} \left[\varphi(\alpha) + (\rho,\alpha) - \frac{1}{2} m(\alpha,\alpha)\right]^{\dim \widetilde{V}(\lambda)_{m\alpha}} \times$$

$$\prod_{\alpha\in\Delta^+_1} \prod_{m=1}^{\infty} \left[\varphi(\alpha) + (\rho,\alpha) - \frac{1}{2}(2m-1)(\alpha,\alpha)\right]^{dim \widetilde{V}(\lambda)_{(2m-1)\alpha}}$$

6.7. **Sketch of the proof of Theorem 6.6.1.** This subsection being very close to the proposition 9.7, [J1], we present only a sketch of the proof. First comes a

6.7.1. **Definition.** We call $\lambda \in \mathfrak{h}^*$ dominant if

$$\begin{array}{ll} \forall\alpha \in \overline{\Delta^+_0}, & \langle\lambda+\rho,\alpha\rangle \notin -\mathbb{N}^+ \\ \forall\alpha \in \Delta^+_1, & \langle\lambda+\rho,\alpha\rangle \notin -2\mathbb{N}-1 \end{array}$$

One has the key

6.7.2. **Lemma.** *The following conditions on λ are equivalent*

(i) $\widetilde{M}(\lambda)$ *is projective in* $\widetilde{\mathcal{O}}$
(ii) $\forall\mu \in \mathfrak{h}^*$, $\mathrm{Hom}_{\mathfrak{g}}(\widetilde{M}(\lambda), \widetilde{M}(\mu)) \neq 0 \Longrightarrow \mu = \lambda$
(iii) λ *is dominant*
(iv) $\forall\mu \in \mathfrak{h}^*$, $[\widetilde{M}(\mu) : \widetilde{V}(\lambda)] \neq 0 \Longrightarrow \mu = \lambda.$

One deduce from the lemma the following

6.7.3. **Proposition.** *Let μ be dominant, $\mu' \in \mathfrak{h}^*$ and $\lambda \in P^+(\pi)$. Then*

$$\left[\mathrm{F}(\widetilde{M}(\mu), \widetilde{M}(\mu')) : \widetilde{V}(\lambda)\right] = \dim \widetilde{V}(\lambda)_{\mu-\mu'}$$

6.7.4. Retain notations of Theorem 6.6.1. Recall that $\mu \in \Lambda_{m,\alpha}$ and hence is dominant. According to factorization of Shapovalov determinants given in Theorem 4.2, $\widetilde{M}(\mu)$ contains a unique copy of $\widetilde{M}(s_\alpha.\mu)$. Moreover we have the short exact sequence

$$0 \longrightarrow \widetilde{M}(s_\alpha.\mu) \longrightarrow \widetilde{M}(\mu) \longrightarrow \widetilde{V}(\mu) \longrightarrow 0$$

and $\widetilde{M}(s_\alpha.\mu)$ is a simple Verma module. An easy consequence of the lemma 6.7.2 is the exactness of the functor $\mathrm{F}(\widetilde{M}(\mu), -)$. This provides the exact sequence

$$0 \longrightarrow \mathrm{F}\Big(\widetilde{M}(\mu), \widetilde{M}(s_\alpha.\mu)\Big) \longrightarrow \mathrm{F}\Big(\widetilde{M}(\mu), \widetilde{M}(\mu)\Big) \longrightarrow \mathrm{F}\Big(\widetilde{M}(\mu), \widetilde{V}(\mu)\Big) \longrightarrow 0$$

One the other hand, the functor $\mathrm{F}\Big(-, \widetilde{V}(\mu)\Big)$ gives the exact sequence

$$\mathrm{F}\Big(\widetilde{M}(s_\alpha.\mu), \widetilde{V}(\mu)\Big) \longleftarrow \mathrm{F}\Big(\widetilde{M}(\mu), \widetilde{V}(\mu)\Big) \longleftarrow \mathrm{F}\Big(\widetilde{V}(\mu), \widetilde{V}(\mu)\Big) \longleftarrow 0$$

An argument from Gelfand-Kirillov dimension theory shows that $\mathrm{F}\Big(\widetilde{M}(s_\alpha.\mu), \widetilde{V}(\mu)\Big) = 0$. The theorem then results from the proposition 6.7.3.

6.8. **"Exotic" factors of PRV determinant.** Call the factors of the PRV determinants "exotic" if they are not standard. By Corollary 6.6.2, the total degree of the product of the standard factors of $\det PRV^\lambda$ is at least

$$\sum_{m=1}^{\infty} \left(\sum_{\alpha \in \overline{\Delta}^+{}_0} \dim \widetilde{V}(\lambda)_{m\alpha} + \sum_{\beta \in \Delta_1^+} \dim \widetilde{V}(\lambda)_{(2m-1)\beta} \right).$$

Recall that $\Delta_1^+ \subset W\beta_1$ and so $\dim \widetilde{V}(\lambda)_{k\beta} = \dim \widetilde{V}(\lambda)_{k\beta_1}$ for all $\beta \in \Delta_1^+$. Taking into account that $\Delta_0^+ = 2\Delta_1^+ \cup \overline{\Delta}^+{}_0$, we conclude from (31) that the total degree of the product of the exotic factors of $\det PRV^\lambda$ is less than or equal to

$$\sum_{\beta \in \Delta_1^+} \sum_{m=1}^{\infty} (-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta}. \tag{32}$$

6.8.1. *Remark.* Consider the case $l = 1$, i.e. $\mathfrak{g} = \mathrm{osp}(1,2)$ and retain notations of 6.3. Then the formula (32) takes the values

$$\sum_{m=1}^{\infty} (-1)^{m+1} \dim \widetilde{V}(n)_{m\beta} = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases} \tag{33}$$

6.9. Let $\mathfrak{p}$ be

$$\mathfrak{p} := \mathfrak{g}_{\pm\beta_l} \oplus \mathfrak{g}_{\pm 2\beta_l} \oplus \mathbb{C}\varphi(\beta_l).$$

Then $\mathfrak{p}$ is a Lie subsuperalgebra of $\mathfrak{g}$ and there is an obvious isomorphism $\mathfrak{p} \cong \mathrm{osp}(1,2)$ which maps the element $\varphi(\beta_l)$ to $\varphi(\beta)$.

We shall denote a simple $\mathfrak{p}$-module of the highest weight $n\beta_l$ by $\breve{V}(n)$ and call a simple $\mathfrak{p}$-module V "odd" if $V \cong \breve{V}(n)$ for some odd number n. Lemma 6.3.4 has the following useful generalization.

6.9.1. **Lemma.** *For any odd* $\mathrm{ad}\,\mathfrak{p}$*-submodule* V *of* $\mathcal{U}(\mathfrak{g})$ *and any* $v \in V$*, the Harish-Chandra projection* $\Upsilon(v)$ *is divisible by* $\varphi(\beta_l) + \frac{1}{2} = (\varphi(\beta_l) + (\rho, \beta_l))$*.*

Proof. Since β_l is a simple root of Δ, $(\mathfrak{p}+\mathfrak{b}^{\pm})$ are Lie subsuperalgebras and even parabolic subsuperalgebras of $\mathfrak{g}$. Let $\mathfrak{h}'$ be the linear span of $\{\varphi(\beta_i)\}_{i=1}^{l-1}$. The superalgebra $(\mathfrak{p}\oplus\mathfrak{h}')$ is the Levi factor of the parabolic superalgebras $(\mathfrak{p}+\mathfrak{b}^{\pm})$. Let $\mathfrak{m}^{\pm}$ be the nilradical of $(\mathfrak{p}+\mathfrak{b}^{\pm})$. Denote by $\mathcal{U}(\mathfrak{p}+\mathfrak{h}')$ the universal enveloping algebra of $(\mathfrak{p}+\mathfrak{h}')$. Let P be the projection of $\mathcal{U}(\mathfrak{g})$ onto $\mathcal{U}(\mathfrak{p}+\mathfrak{h}')$ with respect to decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}+\mathfrak{h}')\oplus(\mathfrak{m}^-\mathcal{U}(\mathfrak{g})+\mathcal{U}(\mathfrak{g})\mathfrak{m}^+)$. Then $\Upsilon = \Upsilon \circ P$. Fix a simple odd $ad\,\mathfrak{p}$-module $V \subset \mathcal{U}(\mathfrak{g})$ and $v \in V$. Since P is an $ad\,\mathfrak{p}$-homomorphism, it follows that either $P(V) = 0$ or $P(V) \simeq V$. In the first case the assertion obviously holds. Consider the second case. Since $\mathfrak{p}$ acts trivially on $\mathfrak{h}'$ and on its universal enveloping algebra $\mathcal{U}(\mathfrak{h}')$, the multiplication map induces an isomorphism $\mathcal{U}(\mathfrak{p}+\mathfrak{h}') = \mathcal{U}(\mathfrak{p})\otimes\mathcal{U}(\mathfrak{h}')$ as $\mathfrak{p}$-modules. Write $P(v) = \sum p_i h_i$ with non-zero $p_i \in \mathcal{U}(\mathfrak{p}),\ h_i \in \mathcal{U}(\mathfrak{h}')$. Since $ad\mathcal{U}(\mathfrak{p})P(v)\simeq V$ is a simple $\mathfrak{p}$-module, one can suppose that $V \simeq ad\mathcal{U}(\mathfrak{p})(p_i)$ for all i. Since $p_i \in \mathcal{U}(\mathfrak{p})$, $\Upsilon(p_i)$ is divisible by $(\varphi(\beta_l) + \frac{1}{2})$ for all i by Lemma 6.3.4. Thus $\Upsilon(v) = \Upsilon(P(v)) = \sum \Upsilon(p_i h_i) = \sum \Upsilon(p_i)h_i$ is divisible by $(\varphi(\beta_l) + \frac{1}{2})$ as required. This establishes the lemma. $\square$

6.9.2. Fix $\lambda \in P^+(\pi)$ and an $ad\,\mathfrak{g}$-submodule $V \cong \widetilde{V}(\lambda)$ of $\mathcal{H}$. Then there is an $ad\,\mathfrak{p}$-submodule of V given by the formula

$$\breve{V} := \bigoplus_{m\in\mathbb{Z}} V_{m\beta_l}.$$

Fix a decomposition of $\breve{V}$ into the sum of simple $ad\,\mathfrak{p}$-modules. Taking into account that $V_0 \subset \breve{V}$ and that the dimension of the zero weight space of any simple $\mathfrak{p}$-module is equal to one, we conclude that the number of simple $ad\,\mathfrak{p}$-modules in the decompositon of $\breve{V}$ is equal to $\dim V_0$. Hence

$$\breve{V} = \bigoplus_{i=1}^{\dim V_0} \breve{V}^{(i)}.$$

Using 6.8.1 we obtain

$$\sum_{m=1}^{\infty} (-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta_l} = \sum_{i=1}^{\dim V_0} \sum_{m=1}^{\infty} (-1)^{m+1} \dim \breve{V}^{(i)}_{m\beta_l} = \#\{i :\ \breve{V}^{(i)} \text{ is odd }\}. \tag{34}$$

Choose a basis $\{v_i\}_{i=1}^{\dim V_0}$ of the subspace V_0 such that $v_i \in \breve{V}^{(i)}$ for all $i = 1, \dots, \dim V_0$. Consider a matrix PRV^{λ} constructed using this basis. By Lemma 6.9.1 the polynomial $\Upsilon(v_i)$ is divisible by $(\varphi(\beta_l) + (\rho, \beta_l))$ if $\breve{V}^{(i)}$ is an odd $\mathfrak{p}$-module. Thus all entries of the i-th line of the matrix PRV^{λ} are divisible by $(\varphi(\beta_l) + (\rho, \beta_l))$ if $\breve{V}^{(i)}$ is an odd $\mathfrak{p}$-module. Using the equality (34) we obtain the

6.9.3. **Corollary.**

(i) For any $\lambda \in P^+(\pi)$ *and any* $\mu \in \mathfrak{h}^*$ *such that* $(\mu+\rho, \beta_l) = 0$ *one has*

$$\operatorname{corank} PRV^\lambda(\mu) \geq \sum_{m=1}^{\infty} (-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta_l}.$$

(ii) For any $\lambda \in P^+(\pi)$*, the determinant* $\det PRV^\lambda$ *is divisible by*

$$(\varphi(\beta_l) + (\rho, \beta_l))^{\sum_{m=1}^{\infty}(-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta_l}}$$

6.10. **Democracy principle.** In Corollary 6.9.3 we found $m_\lambda := \sum_{m=1}^{\infty}(-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta_l}$ linear exotic factors of $\det PRV^\lambda$. By (32) the number of linear exotic factors is at most $l \cdot m_\lambda$. In order to obtain the rest of the factors we are going to prove in this subsection the following democracy theorem establishing "equality of the odd positive roots according Law".

Theorem.

(i) For any $\lambda \in P^+(\pi)$ *and any* $\mu \in \mathfrak{h}^*$ *such that* $(\mu+\rho, \beta_i) = 0$ *one has*

$$\operatorname{corank} PRV^\lambda(\mu) \geq \sum_{m=1}^{\infty} (-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta_l}.$$

(ii) For any $\lambda \in P^+(\pi)$, $\beta_i \in \Delta_1^+$ *the determinant* $\det PRV^\lambda$ *is divisible by*

$$(\varphi(\beta_i) + (\rho, \beta_i))^{\sum_{m=1}^{\infty}(-1)^{m+1} \dim \widetilde{V}(\lambda)_{m\beta_l}}.$$

6.10.1. Let $s_i \in W$ be the simple reflection with respect to the simple root α_i, where $\alpha_i = (\beta_i - \beta_{i+1})$ for $i < l$ and $\alpha_l = \beta_l$. In order to prove the theorem above we need the following lemma.

Lemma. *Fix* $i \in \{1, \dots, l-1\}$ *and* $n \in \mathbb{N}^+$*. Take* $\mu \in \mathfrak{h}^*$ *such that*

$$(\beta_i, \mu+\rho) = n, \ (\beta_{i+1}, \mu+\rho) = 0,$$
$$\text{the collection} \{(\beta_k, \mu+\rho)\}_{k \neq i+1} \ \text{ is linearly independent over } \mathbb{Q}.$$

(i) Let N *be the kernel of the surjective map* $\widetilde{M}(\mu) \to \widetilde{V}(\mu)$*. Then*

$$\min_{\xi \in \Omega(N)} (\mu - \xi, \omega_i) = n.$$

(ii) Assume that $\lambda \in P^+(\pi)$ *is such that* $(\lambda, \omega_i) < n$*. Then*

$$\overline{C}(\lambda) \cap \operatorname{Ann} \widetilde{V}(\mu) \subseteq \overline{C}(\lambda) \cap \operatorname{Ann} \widetilde{V}(s_i.\mu).$$

where $\overline{C}(\lambda)$ *is the isotypical component of* $\widetilde{V}(\lambda)$ *in* $\mathcal{U}(\mathfrak{g})$.

Proof. (i) The assumption on μ implies that $\langle \mu+\rho,\alpha\rangle \in \mathbb{N}^+$ only when $\alpha=\beta_i \pm \beta_{i+1}$ or $\alpha=\beta_i$. Since $\langle \mu+\rho,\beta_i\rangle = 2n \notin 2\mathbb{N}+1$ and $\langle \mu+\rho,\beta_i\pm\beta_{i+1}\rangle = n$, we conclude from 4.3 that

$$\mu-\Omega(N)=\{n(\beta_i-\beta_{i+1})+\mathbb{N}(\pi)\}\cup\{n(\beta_i+\beta_{i+1})+\mathbb{N}(\pi)\}.$$

Hence

$$\min_{\xi\in\Omega(N)}(\mu-\xi,\omega_i)=\min_{\nu\in\mathbb{N}(\pi)}\{(n(\beta_i-\beta_{i+1})+\nu,\omega_i),(n(\beta_i+\beta_{i+1})+\nu,\omega_i)\}=n$$

as required.

(ii) Since $(\mu+\rho,\beta_i-\beta_{i+1})\in\mathbb{N}^+$ it follows that $\widetilde{V}(s_i.\mu)$ is a subquotient of $\widetilde{M}(\mu)$. Therefore $\operatorname{Ann}\widetilde{M}(\mu)\subseteq\operatorname{Ann}\widetilde{V}(s_{\alpha_i}.\mu)$. Thus it is sufficient to show that

$$\overline{C}(\lambda)\cap\operatorname{Ann}\widetilde{V}(\mu)=\overline{C}(\lambda)\cap\operatorname{Ann}\widetilde{M}(\mu).$$

Assume the contrary, namely, that there exists an $ad\,\mathfrak{g}$-submodule $V\cong\widetilde{V}(\lambda)$ of $\operatorname{Ann}\widetilde{V}(\mu)$ such that $V\widetilde{M}(\mu)\neq 0$. Since V is $ad\,\mathfrak{g}$-invariant one has $V\mathcal{U}(\mathfrak{g})=\mathcal{U}(\mathfrak{g})V$. Consequently, $V\widetilde{M}(\mu)\neq 0$ forces $Vv_\mu\neq 0$, where v_μ is a highest weight vector of $\widetilde{M}(\mu)$. Recall from 2.6.1 that $\Omega(V)=-\Omega(V)$. Therefore

$$\max_{\xi\in\Omega(Vv_\mu)}(\mu-\xi,\omega_i)=\max_{\nu\in\Omega(V)}(-\nu,\omega_i)=\max_{\nu\in\Omega(V)}(\nu,\omega_i)<n. \tag{35}$$

On the other hand, $V\widetilde{V}(\mu)=0$ and so $Vv_\mu\subseteq N$. Thus (35) contradicts to (i). $\square$

6.10.2. *Proof of Theorem 6.10.* Fix $\lambda\in P^+(\pi)$ and set

$$r=\sum_{m=1}^{\infty}(-1)^{m+1}\dim\widetilde{V}(\lambda)_{m\beta_l}.$$

For $k=1,\dots,l$ let x_k be the element of $\mathcal{S}(\mathfrak{h})$ given by the formula $x_k:=\varphi(\beta_k)+(\rho,\beta_k)$. Recall that s_i, $i=1,\dots,l-1$, is the simple reflection with respect to the root $(\beta_i-\beta_{i+1})$ and consequently one has, for any $i=1,\dots,l-1$

$$x_i(s_i.\mu)=x_{i+1}(\mu),\ \ x_{i+1}(s_i.\mu)=x_i(\mu),\ \ \ x_k(s_i.\mu)=x_k(\mu)\text{ for }k\neq i,i+1. \tag{36}$$

Each entry $a_{i,j}$ of the matrix PRV^λ is an element of $\mathcal{S}(\mathfrak{h})$ and so can be considered as a polynomial in $x_1,\dots,x_l$. We have to prove that for any $i=1,\dots,l$ and for any μ such that $x_i(\mu)=0$, the corank of the matrix $PRV^\lambda(x_1(\mu),\dots,x_l(\mu))$ is at least r. We prove this assertion by induction on i. By Corollary 6.9.3, the assertion holds for $i=l$. Assume that the assertion holds for some $i+1:\ 1<i<l$ and deduce it for i. Recall Claim 6.2.1:

$$[\operatorname{Ann}_{\mathcal{H}}\widetilde{V}(\mu):\widetilde{V}(\lambda)]=\operatorname{corank}PRV^\lambda(\mu) \tag{37}$$

Thus our assumption implies that for any μ such that $(\mu+\rho,\beta_{i+1})=0$ one has

$$[\operatorname{Ann}_{\mathcal{H}}\widetilde{V}(\mu):\widetilde{V}(\lambda)]\geq r. \tag{38}$$

Fix, for a moment, a collection of complex numbers $\{a_k\}_{k=1}^l$ such that the collection $\{1,a_k: k=1,\dots,l\}$ is linearly independent over $\mathbb{Q}$. For every $n\in\mathbb{N}^+$, $n>(\lambda,\omega_i)$, take $\mu_n\in\mathfrak{h}^*$

such that $x_i(\mu_n)=n,\ x_{i+1}(\mu_n)=0,\ x_k(\mu_n)=a_k$ for $k\neq i,i+1$. Then Lemma 6.10.1 (ii) combined with the inequality (38) implies that

$$\begin{aligned} r\leq [\operatorname{Ann}_{\mathcal{H}}\widetilde{V}(s_i.\mu_n):\widetilde{V}(\lambda)] &= \operatorname{corank} PRV^{\lambda}(s_i.\mu_n) \\ &= \operatorname{corank} PRV^{\lambda}(x_1(s_i.\mu_n),\dots,x_l(s_i.\mu_n)). \end{aligned}$$

Thus $\operatorname{corank} PRV^{\lambda}$ is greater than or equal to r at the points

$$(x_1(s_i.\mu_n),\dots,x_l(s_i.\mu_n))=(a_1,\dots,a_{i-1},0,n,a_{i+2},\dots,a_l) \quad \text{(see (36))}$$

of the line

$$(a_1,\dots,a_{i-1},0,\mathbb{C},a_{i+2},\dots,a_l).$$

Therefore $\operatorname{corank} PRV^{\lambda}\geq r$ on the whole line $(a_1,\dots,a_{i-1},0,\mathbb{C},a_{i+2},\dots,a_l)$. Observe that the union of the lines $(a_1,\dots,a_{i-1},0,\mathbb{C},a_{i+2},\dots,a_l)$ such that $\{1,a_k:k=1,\dots,l\}$ is linearly independent over $\mathbb{Q}$ is dense in the hyperplane $x_i=0$. Hence $\operatorname{corank} PRV^{\lambda}$ is greater than or equal to r in the whole hyperplane $x_i=0$ as required. This establishes Theorem 6.10.

6.11. Combining the results of Corollary 6.6.2, Theorem 6.10 and the bound (32) we obtain Theorem 6.5.

7. Main theorem

7.1. **Theorem.** *Take $\lambda\in\mathfrak{h}^*$. Then*

$$\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})}\widetilde{M}(\lambda)=\mathcal{U}(\mathfrak{g})\operatorname{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}(\lambda) \iff \forall\alpha\in\Delta_1^+:\ (\lambda+\rho,\alpha)\neq 0. \tag{39}$$

The standard factors of Theorem 6.5 are exactly the factors which appear in the factorization of Shapovalov determinants—see Theorem 4.2. Since a Verma module $\widetilde{M}(\mu)$ is simple iff μ is not a root of any Shapovalov determinant, this proves the equivalence (39) for simple Verma modules. By 3.2, it gives immediatly the implication "$\Leftarrow$" of (39).

7.2. Let us prove the other implication. Consider $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module. All its weight-spaces are one-dimensional, $\Omega(\mathfrak{g}_1)=\{\pm\beta_1,\dots,\pm\beta_l\}$ so $\mathfrak{g}_1\cong V(\omega_1)$. Consider the natural representation $\widetilde{V}(\omega_1)$ of $\operatorname{osp}(1,2l)$. It is easy to see that $\widetilde{V}(\omega_1)=V(\omega_1)\oplus V(0)$ as $\mathfrak{g}_0$-modules. Again all weight-spaces of $\widetilde{V}(\omega_1)$ are one-dimensional and $\Omega(\widetilde{V}(\omega_1))=\{0,\pm\beta_1,\dots,\pm\beta_l\}$. Hence, by Theorem 5.4, $\mathcal{H}$ contains exactly one copy V of $\widetilde{V}(\omega_1)$ which occurs in degree $2l$. Fix $i\in\{1,\dots,l\}$. One has

$$\sum_{m=1}^{\infty}(-1)^{m+1}\dim\widetilde{V}(\omega_1)_{m\beta_i}=1$$

and so, by Theorem 6.5, $\det PRV^{\omega_1}(\mu) = 0$ for any $\mu \in \mathfrak{h}^*$ such that $(\mu+\rho, \beta_i) = 0$. By Lemma 6.2.1, it implies that $V \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{V}(\mu)$. In other words, $V \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{M}(\mu)$ for any μ in the set

$$\left\{\mu \in \mathfrak{h}^* | \ (\mu+\rho, \beta_i) = 0 \text{ and } \widetilde{M}(\mu) \text{ is simple}\right\}.$$

According to the criterion of simplicity given in Corollary 4.4, this set is Zariski dense in the hyperplane $(\mu+\rho, \beta_i) = 0$.

The following lemma proves that in fact $V \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{M}(\mu)$ for any μ such that $(\mu + \rho, \beta_i) = 0$. This gives the implication "$\Longrightarrow$" of (39).

7.3. **Lemma.** *Let V be an* ad*-invariant subspace of $\mathcal{U}(\mathfrak{g})$. Then the set* $\left\{\lambda \in \mathfrak{h}^*,\ V \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{M}(\lambda)\right\}$ *is a Zariski closed subset of $\mathfrak{h}^*$.*

Proof. For any $\lambda \in \mathfrak{h}^*$ denote by v_λ a highest weight vector of $\widetilde{M}(\lambda)$. Remark that the *ad*-invariance of V implies that $\mathcal{U}(\mathfrak{g})V = V\mathcal{U}(\mathfrak{g})$. Hence, for any $\lambda \in \mathfrak{h}^*$, one has the equivalence $V \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{M}(\lambda) \Longleftrightarrow V.v_\lambda = 0$. Let $\{a_j\}_{j\in J}$ be a basis of V. Then

$$\left\{\lambda \in \mathfrak{h}^*,\ V \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{M}(\lambda)\right\} = \bigcap_{j \in J} \left\{\lambda \in \mathfrak{h}^*,\ x.v_\lambda = 0\right\}.$$

Thus it is sufficient to show that for every $a \in \mathcal{U}(\mathfrak{g})$, the set $\{\lambda \in \mathfrak{h}^*,\ a.v_\lambda = 0\}$ is a Zariski closed subset of $\mathfrak{h}^*$. Using the triangular decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^+)$, write $a = y_1P_1 + \ldots + y_rP_r + x$ where $y_1, \ldots, y_r$ are linearly independent elements of $\mathcal{U}(\mathfrak{n}^-)$, $P_1, \ldots, P_r \in \mathcal{U}(\mathfrak{h})$ and $x \in \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$. Then

$$\begin{aligned} a.v_\lambda = 0 &\iff \Big(\sum_i y_iP_i\Big).v_\lambda = 0 \\ &\iff \sum_i P_i(\lambda) y_i.v_\lambda = 0 \ . \\ &\iff \forall i \ P_i(\lambda) = 0. \end{aligned}$$

The lemma is proved. □

This completes the proof of Theorem 7.1.

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